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Notes on Relative Motion.

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1. Motion of a point in a plane.

At time t let the moving axes be $O\xi$, $O\eta$, and P a point (ξ, η) in their plane. At time $t + \delta t$ let these axes coincide with $O\xi'$, $O\eta'$, and P with P'; then the ξ and η components of the displacement PP' are $-\omega\eta\delta t$, $\omega\xi\delta t$, respectively, if ω is the rate at which the axes turn round $O\zeta$. Let a moving point be at P at time t, and at Q at time $t + \delta t$, the co-ordinates of Q referred to $O\xi'$, $O\eta'$ being $\xi + \dot{\xi}\delta t$, $\eta + \dot{\eta}\delta t$; then the absolute velocity of the moving point is ultimately $\frac{PQ}{\delta t} = \left(\frac{PP'}{\delta t}, \frac{P'Q}{\delta t}\right)$, the ξ and η components of which are $\dot{\xi} - \omega\eta$, $\dot{\eta} + \omega\xi$, respectively.

Putting $\dot{\xi} - \omega\eta = u = OA$, and $\dot{\eta} + \omega\xi = v = OB$, the component velocities at time $t + \delta t$ become $u + \dot{u}\delta t = OA'$ along $O\xi'$, and $v + \dot{v}\delta t = OB'$ along $O\eta'$. Hence the absolute acceleration ultimately $= \left(\frac{AA'}{\delta t}, \frac{BB'}{\delta t}\right)$, the components of which are

$$\dot{u} - v\omega = \ddot{\xi} - 2\,\omega\dot{\eta} - \eta\dot{\omega} - \omega^2\xi \text{ along } O\xi,$$

$$\dot{v} + u\omega = \ddot{\eta} + 2\,\omega\dot{\xi} + \xi\dot{\omega} - \omega^2\eta \text{ along } O\eta.$$

2. Motion of a rigid body round a fixed axis $O\zeta$, the axes $O\xi$, $O\eta$ being fixed in the body.

At time t the whole momentum is $-M\omega\eta = OA$ along $O\xi$, and $M\omega\xi = OB$ along $O\eta$, where ξ, η are co-ordinates of the centre of inertia. At time $t + \delta t$ the momentum is $-M\eta\,(\omega + \dot{\omega}\delta t) = OA'$ along $O\xi'$, and $M\xi\,(\omega + \dot{\omega}\delta t) = OB'$ along $O\eta'$. The changes of momentum per unit time are, therefore, ultimately $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}$, whose components are

$$-M\eta\dot{\omega} - M\omega^2\xi \text{ along } O\xi,$$

$$M\xi\dot{\omega} - M\omega^2\eta \text{ along } O\eta.$$

At time t the whole moment of momentum is (employing OA, OB in a new sense)

$$-\beta\omega = OA \text{ along } O\xi,$$

$$-\alpha\omega = OB \text{ along } O\eta,$$

$$C\omega \;.\;.\;.\;. \text{ along } O\zeta,$$

where $\alpha = \Sigma m\eta\zeta, \quad C = \Sigma m(\xi^2 + \eta^2)$, etc.

At time $t + \delta t$ the moment of momentum becomes

$$-\beta(\omega + \dot{\omega}\delta t) = OA' \text{ along } O\xi',$$

$$-\alpha(\omega + \dot{\omega}\delta t) = OB' \text{ along } O\eta', \text{ etc.}$$

Hence the changes per unit time of moment of momentum are ultimately $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, C\dot{\omega}$, the components of which are $-\beta\dot{\omega} + \alpha\omega^2$ along $O\xi$, $-\alpha\dot{\omega} - \beta\omega^2$ along $O\eta$, and $C\dot{\omega}$ along $O\zeta$.

These, it will be observed, are of the same form as when the axes are fixed in space.

3. To measure the absolute velocity and acceleration of a point referred to axes moving in space round O.

Let the motion of the axes be due to rotations $\theta_1, \theta_2, \theta_3$ measured along themselves. Then, proceeding as in § 1, the displacements of a point $P(\xi, \eta, \zeta)$ due to these rotations are $(\zeta\theta_2 - \eta\theta_3)\,\delta t$ along $O\xi$, $(\xi\theta_3 - \zeta\theta_1)\,\delta t$ along $O\eta$, and $(\eta\theta_1 - \xi\theta_2)\,\delta t$ along $O\zeta$. These added to the relative displacements $(\dot{\xi}\delta t, \dot{\eta}\delta t, \dot{\zeta}\delta t)$ of the moving point give the absolute displacements. Hence the components of the absolute velocity are

$$u = OA = \dot{\xi} + \zeta\theta_2 - \eta\theta_3 \text{ along } O\xi,$$

$$v = OB = \dot{\eta} + \xi\theta_3 - \zeta\theta_1 \text{ along } O\eta,$$

$$w = OC = \dot{\zeta} + \eta\theta_1 - \xi\theta_2 \text{ along } O\zeta.$$

Again, let the velocities at time $t + \delta t$ be $OA' = u + \dot{u}\delta t$ along $O\xi'$, etc.; then the absolute accelerations are ultimately $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t}$, whose components are

$$\dot{u} - v\theta_3 + w\theta_2 \text{ along } O\xi,$$

$$\dot{v} - w\theta_1 + u\theta_3 \text{ along } O\eta,$$

$$\dot{w} - u\theta_2 + v\theta_1 \text{ along } O\zeta.$$

These become, on reduction,

$$\ddot{\xi} - 2\,\theta_3\dot{\eta} + 2\,\theta_2\dot{\zeta} + \zeta\dot{\theta}_2 - \eta\dot{\theta}_3 - (\theta_1^2 + \theta_2^2 + \theta_3^2)\,\xi + (\xi\theta_1 + \eta\theta_2 + \zeta\theta_3)\,\theta_1$$

along $O\xi$, etc.

NOTE. — These resolutions are most readily effected as follows: AA' is equivalent to AD along $O\eta$, DH along $O\zeta$, and HA' along $O\xi'$; and similar resolutions are effected for BB', CC'. The values of AD, DH, etc. are at once derived from the displacements in time δt of the points (1, 0, 0), (0, 1, 0), (0, 0, 1). The latter are, respectively,

$$\begin{array}{ccc} 0\ , & \theta_3\ , & -\theta_2\ , \\ -\theta_3\ , & 0\ , & \theta_1\ , \\ \theta_2\ , & -\theta_1\ , & 0\ , \end{array}$$

each multiplied by δt; from which the values of AD, DH, etc. are obtained by multiplying the first set by OA, the second by OB, and the third by OC. Moreover, the parts HA', etc. remain unchanged in magnitude when resolved along $O\xi$, $O\eta$, $O\zeta$, if infinitesimals above the first order be neglected. Thus, in the present case, $HA' = \dot{u}\delta t$, $AD = u\theta_3\delta t$, $DH = -u\theta_2\delta t$.

4. If, in the previous case, the origin moves, its acceleration must of course be added to the expressions found in § 3. These formulas may be tested by the following well-known example. Let O be on the earth's surface in latitude λ, and let $O\xi$ be drawn south, $O\eta$ east, and $O\zeta$ vertical. Then ω being the earth's rotation and r its radius, the accelerations of O are

$$\begin{array}{lll} -\,\omega^2 r\cos\lambda\sin\lambda & \text{along} & O\xi\,, \\ -\,\omega^2 r\cos^2\lambda & \text{``} & O\zeta\,. \end{array}$$

Also, $\theta_1 = -\,\omega\cos\lambda$, $\theta_2 = 0$, $\theta_3 = \omega\sin\lambda$, and $\dot{\theta}_1 = 0 = \dot{\theta}_2 = \dot{\theta}_3$.

Hence the accelerations of m at (ξ, η, ζ) are

$$\begin{array}{l} \ddot{\xi} - \omega^2 r\cos\lambda\sin\lambda - 2\,\omega\dot{\eta}\sin\lambda - \omega^2\xi\sin^2\lambda - \omega^2\zeta\sin\lambda\cos\lambda, \\ \ddot{\eta} + 2\,\omega\dot{\zeta}\cos\lambda + 2\,\omega\dot{\xi}\sin\lambda - \omega^2\eta, \\ \ddot{\zeta} - \omega^2 r\cos^2\lambda - 2\,\omega\dot{\eta}\cos\lambda - \omega^2\zeta\cos^2\lambda - \omega^2\xi\sin\lambda\cos\lambda, \end{array}$$

along $O\xi$, $O\eta$, $O\zeta$, respectively.

5. To measure the changes in the rotation of a rigid body with one point fixed, the axes moving as in § 3. Let the rotations to which the displacement of the body is due be at time t, $\omega_1 = OA$, $\omega_2 = OB$, $\omega_3 = OC$ measured respectively along $O\xi$, $O\eta$, $O\zeta$. Then since at time $t + \delta t$ these become $\omega_1 + \dot{\omega}_1\delta t = OA'$, etc. along $O\xi'$, $O\eta'$, $O\zeta'$, the absolute changes per unit time in the rotation are ultimately

$$\frac{AA'}{\delta t}\,,\ \frac{BB'}{\delta t}\,,\ \frac{CC'}{\delta t}\,.$$

Resolving these, we get for the required components

$$\dot{\omega}_1 - \omega_2\theta_3 + \omega_3\theta_2 \text{ along } O\xi\text{, etc.}$$

6. To measure the change in the whole absolute momentum of a rigid body, one point of which is fixed at O, the axes moving as in §§ 3, 5.

Since the absolute momentum of m in the position (ξ, η, ζ) at time t is

$$m\,\{\zeta(\omega_2+\theta_2)-\eta(\omega_3+\theta_3)\} \text{ along } O\xi, \text{ etc.},$$

it follows that the whole absolute momentum at that time is

$$\begin{aligned} &z(\omega_2+\theta_2)-y(\omega_3+\theta_3) \text{ along } O\xi,\\ &x(\omega_3+\theta_3)-z(\omega_1+\theta_1) \text{ along } O\eta,\\ &y(\omega_1+\theta_1)-x(\omega_2+\theta_2) \text{ along } O\zeta,\end{aligned}$$

each multiplied by M, where (x, y, z) is the position of the centre of inertia. Calling these components $\mu_1 = OA$, $\mu_2 = OB$, $\mu_3 = OC$, respectively, it follows that at time $t+\delta t$ they become $\mu_1+\dot\mu_1\,\delta t = OA'$ along $O\xi'$, $\mu_2+\dot\mu_2\,\delta t = OB'$ along $O\eta'$, $\mu_3+\dot\mu_3\delta t = OC'$ along $O\zeta'$. The changes in the whole momentum per unit time are, therefore, $\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t}$, whose components are

$$\begin{aligned} &\dot\mu_1-\mu_2\,\theta_3+\mu_3\,\theta_2 \text{ along } O\xi,\\ &\dot\mu_2-\mu_3\,\theta_1+\mu_1\,\theta_3 \text{ along } O\eta,\\ &\dot\mu_3-\mu_1\,\theta_2+\mu_2\,\theta_1 \text{ along } O\zeta.\end{aligned}$$

Since $\dot x = z\omega_2 - y\omega_3$, etc. these expressions become, on reduction, M times

$$\begin{aligned} z(\dot\omega_2+\dot\theta_2)-y(\dot\omega_3+\dot\theta_3)+\omega_1\,\{(\omega_1+\theta_1)\,x+(\omega_2+\theta_2)\,y+(\omega_3+\theta_3)z\}\\ +(\omega_1+\theta_1)\,(\theta_1\,x+\theta_2\,y+\theta_3\,z)-x\,\{(\omega_1+\theta_1)^2+(\omega_2+\theta_2)^2+(\omega_3+\theta_3)^2\}\end{aligned}$$

for the first, with similar values for the other two.

7. To measure the changes in the whole absolute moment of momentum under the same circumstances as in § 6. Since the absolute moment of m's momentum at time t is m times

$$(\omega_1+\theta_1)\,(\eta^2+\zeta^2)-(\omega_2+\theta_2)\,\xi\eta-(\omega_3+\theta_3)\,\zeta\xi \text{ along } O\xi,$$

with corresponding components along $O\eta$, $O\zeta$, it follows that the components of the whole moment of momentum at that time are

$$\begin{aligned} &A(\omega_1+\theta_1)-\gamma(\omega_2+\theta_2)-\beta(\omega_3+\theta_3) \text{ along } O\xi,\\ &-\gamma(\omega_1+\theta_1)+B(\omega_2+\theta_2)-\alpha(\omega_3+\theta_3) \text{ along } O\eta,\\ &-\beta(\omega_1+\theta_1)-\alpha(\omega_2+\theta_2)+C(\omega_3+\theta_3) \text{ along } O\zeta,\end{aligned}$$

where $\qquad A = \Sigma m\,(\eta^2+\zeta^2)\,,\quad \alpha = \Sigma m\eta\zeta\,,$ etc.

Let these components be called $\nu_1 = OA$, $\nu_2 = OB$, $\nu_3 = OC$, respectively. Then at time $t + \delta t$ they become $\nu_1 + \dot{\nu}_1\delta t = OA'$ along $O\xi'$, $\nu_2 + \dot{\nu}_2\delta t = OB'$ along $O\eta'$, and $\nu_3 + \dot{\nu}_3\delta t = OC'$ along $O\zeta'$. Hence the changes of the moment of momentum per unit time are

$$\frac{AA'}{\delta t}, \frac{BB'}{\delta t}, \frac{CC'}{\delta t},$$

whose components are

$$\begin{aligned} &\dot{\nu}_1 - \nu_2\theta_3 + \nu_3\theta_2 \text{ along } O\xi, \\ &\dot{\nu}_2 - \nu_3\theta_1 + \nu_1\theta_3 \text{ along } O\eta, \\ &\dot{\nu}_3 - \nu_1\theta_2 + \nu_2\theta_1 \text{ along } O\zeta. \end{aligned}$$

Now, since $\dot{\xi} = \zeta\omega_2 - \eta\omega_3$, etc., it follows that

$$\begin{aligned} \dot{A} &= 2\,\Sigma m\,(\eta\dot{\eta} + \zeta\dot{\zeta}) \\ &= 2\,(\gamma\omega_3 - \beta\omega_2) \\ \dot{B} &= 2\,(\alpha\omega_1 - \gamma\omega_3) \\ \dot{C} &= 2\,(\beta\omega_2 - \alpha\omega_1) \\ \dot{\alpha} &= \Sigma m\,(\eta\dot{\zeta} + \zeta\dot{\eta}) \\ &= (C - B)\,\omega_1 - \gamma\omega_2 + \beta\omega_3 \\ \dot{\beta} &= \gamma\omega_1 + (A - C)\,\omega_2 - \alpha\omega_3 \\ \dot{\gamma} &= -\beta\omega_1 + \alpha\omega_2 + (B - A)\omega_3. \end{aligned}$$

Hence the above values for the component changes of moment of momentum become

$$A\,(\dot{\omega}_1 + \dot{\theta}_1) - \gamma\,(\dot{\omega}_2 + \dot{\theta}_2) - \beta\,(\dot{\omega}_3 + \dot{\theta}_3) + 2\,(\omega_1 + \theta_1)\,(\gamma\omega_3 - \beta\omega_2) - (\omega_2 + \theta_2) \\ [-\beta\omega_1 + \alpha\omega_2 + (B - A)\,\omega_3] - (\omega_3 + \theta_3)\,[\gamma\omega_1 + (A - C)\,\omega_2 - \alpha\omega_3] - \theta_3 \\ [-\gamma\,(\omega_1 + \theta_1) + B\,(\omega_2 + \theta_2) - \alpha\,(\omega_3 + \theta_3)] + \theta_2\,[-\beta\,(\omega_1 + \theta_1) - \alpha \\ (\omega_2 + \theta_2) + C\,(\omega_3 + \theta_3)]$$

for the first; with similar expressions for the other two.